Faraday Rotation of the Cosmic Microwave Background Polarization and Primordial Magnetic Field Properties

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ABSTRACT

Measurements of the Faraday rotation of the cosmic microwave background radiation (CMBR) polarization could provide evidence for the existence of primordial magnetic fields. The Faraday rotation could also allow the study of some properties of these fields. In this paper, we calculate the angular dependence of the Faraday rotation correlator for different assumptions about the spectral index and correlation length of the magnetic field. We show that the helical part of the magnetic field does not make any contribution to the correlator. We stress the importance of the angular resolution of the detector in the Faraday rotation measure, showing that it could severely reduce the effect, even for a relatively large magnetic field correlation length.

Subject headings: magnetic fields–cosmology:theory–cosmic microwave background

1. Introduction

Astronomical observations have revealed the presence of large-scale magnetic fields in the Universe. They exist in all types of galaxies (spiral, elliptical, barred, and irregular), in galaxy clusters, and probably in superclusters, with correlation lengths $\xi \sim$ Mpc and intensities $B \sim \mu$G. The presence of magnetic fields in all gravitationally bound large-scale structures could suggest that they have been created after structure formation through

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astrophysical mechanisms (as, for example, the "Biermann battery"). On the other hand, the detection of magnetic fields in galaxies at high redshifts, could represent a strong hint that magnetic fields have been generated in the early Universe (*i.e* before structure formation) by microphysics processes (for a full discussion see Grasso & Rubinstein 2001; Widrow 2003; Dolgov 2001; Giovannini 2004).

If large-scale magnetic fields have a primordial origin, they could have observable effects on the cosmic microwave background radiation (CMBR). In particular, as discussed in (Kosowsky & Loeb 1996), existence of a magnetic field at the last scattering surface, corresponding to a present-day value of $B_0 \sim 10^{-9}$G, may induce measurable Faraday rotation of the CMBR polarization. Larger magnetic fields, at the level $B_0 \sim 10^{-8}$G, may even depolarize the CMBR (Harari *et al.* 1997), owing to differential Faraday rotation across the last scattering surface.

Clearly the measure of a non-zero Faraday rotation in the CMBR polarization would be extremely important, since it would support the primordial origin of cosmic magnetic fields, and would provide direct information about the physics of the early Universe.

In this paper, we analyze the Faraday rotation of the CMBR polarization induced by a small ($B_0 \sim 10^{-9}$G) primordial stochastic magnetic field. In particular, we calculate the angular dependence of the Faraday rotation measure (RM) maps for different assumptions about the magnetic field spectral index and correlation length. We consider the possibility of helical primordial magnetic fields, showing that, contrary to the suggestion of (Pogosian *et al.* 2002; Pogosian *et al.* 2003), the helical part of the magnetic field does not contribute to the Faraday RM maps at any angle. Finally, we discuss the importance of the detector angular resolution in the Faraday rotation measure, showing that it could severely constrain the actual possibility of observing the effect.

The plan of the paper is as follows. In Section 2, we briefly review the properties of a stochastic homogeneous and isotropic magnetic field and we define its characteristic properties. In Section 3, we discuss the Faraday rotation effects and show that Faraday RM maps do not depend on the helical part of the magnetic field. In Section 4, we discuss the angular dependence of Faraday RM maps for different assumptions on the magnetic field spectrum. Finally, we summarize main results in Section 5.

2. The magnetic field spectrum

We consider primordial stochastic magnetic field $\mathbf{B}(\eta, \mathbf{x})$, created before the matter-radiation decoupling. Its Fourier transform, $\mathbf{B}(\eta, \mathbf{k})$, is defined according to [1]

$$B_i(\eta, \mathbf{k}) = \int d^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, B_i(\eta, \mathbf{x})\,, \qquad B_i(\eta, \mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, B_i(\eta, \mathbf{k})\,. \tag{1}$$

Here η is the conformal time, $\mathbf{x}$ are comoving coordinates, and $\mathbf{k}$ are the comoving wavenumbers.

Under assumption of flux conservation, the magnetic field scales as a^{-2}, where a is the cosmological scale factor (see, *e.g.*, Grasso & Rubinstein 2001). The magnetic field at arbitrary time can thus be related to its present value by $\mathbf{B}(\eta, \mathbf{x}) = \mathbf{B}_0(\mathbf{x})/a(\eta)^2$, where the subscript 0 indicates today's values and we normalize the scale factor according to $a(\eta_0) = 1$.

We are interested in the effects of a statistically homogeneous and isotropic magnetic field. This means that the correlation tensor of the magnetic field, $C_{ij}(\mathbf{r}_1, \mathbf{r}_2) \equiv \langle B_{i0}(\mathbf{r}_1) B_{j0}(\mathbf{r}_2)\rangle$, is a function of $r = |\mathbf{r}_1 - \mathbf{r}_2|$ only and, moreover, it transforms as an $SO(3)$ tensor. In terms of the Fourier amplitudes of the field, these conditions (together with the fact that the magnetic field is a divergence-free field) translate into (see *e.g.* Monin & Yaglom 1975; Caprini *et al.* 2004)

$$\langle B_{i0}(\mathbf{k}) B_{j0}(\mathbf{k}')\rangle = \frac{(2\pi)^3}{2}\delta(\mathbf{k} + \mathbf{k}')[P_{ij}S(k) + i\varepsilon_{ijl}\hat{k}_l A(k)], \tag{2}$$

where $P_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j$, ε_{ijl} is the totally antisymmetric tensor and $\hat{k}_i = k_i/k$. Here $S(k)$ denotes the symmetric part and $A(k)$ denotes the antisymmetric part of the correlator. Usually $S(k)$ is referred to as the *magnetic power spectrum*, being related to the total magnetic energy $E_B = \frac{1}{2}\int d^3x\, \mathbf{B}_0^2(\mathbf{x})$ (the volume is normalized to $V = 1$) by

$$\mathcal{E}_B(k) = 2\pi k^2 S(k)\,, \qquad E_B = \int_0^\infty dk\, \mathcal{E}_B(k)\,. \tag{3}$$

On the other hand, $A(k)$ is referred to as the *helical power spectrum*, being related to the magnetic helicity $H_B = \int d^3x\, \mathbf{A}_0(\mathbf{x}) \cdot \mathbf{B}_0(\mathbf{x})$ by (see *e.g.* Vachaspati 2001):

$$\mathcal{H}_B(k) = 4\pi k A(k)\,, \qquad H_B = \int_0^\infty dk\, \mathcal{H}_B(k)\,. \tag{4}$$

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[1] Here and in the following we use the natural system of units in which $\hbar = c = 1$. We use bold characters to indicate vectors, *i.e.* $\mathbf{a} \equiv \overrightarrow{a}$, while normal characters are used to indicate vector moduli, *i.e.* $a = |\mathbf{a}|$.

We will assume that both $S(k)$ and $A(k)$ can be represented by the following simple functions:

$$S(k) = S_0 k^{n_S} e^{-(k/K)^2}, \qquad A(k) = A_0 k^{n_A} e^{-(k/K)^2}, \tag{5}$$

which for $k \ll K$ possess a power law behavior. For large k, the spectrum is instead suppressed exponentially in order to have finite energy and helicity and a finite correlation length ξ, given by:

$$\xi \equiv \frac{\int_0^\infty dk\,(2\pi/k)\,\mathcal{E}_B(k)}{\int_0^\infty dk\,\mathcal{E}_B(k)} = \frac{2\pi}{K}\frac{\Gamma[(n_S+2)/2]}{\Gamma[(n_S+3)/2]}\,. \tag{6}$$

The two functions $S(k)$ and $A(k)$ are not completely independent, since any field configuration has to satisfy:

$$S(k) \geq |A(k)|. \tag{7}$$

Moreover, requiring that the function (2) is analytic, one can show that the spectral index n_S has to be even ≥ 2, while n_A has to be odd and ≥ 2 (Durrer & Caprini 2003).

3. Faraday Rotation

As discussed in (Kosowsky & Loeb 1996), a primordial magnetic field could leave significant imprints upon the CMBR polarization through the effect of the Faraday rotation. Any magnetic field between the last scattering surface and the observer would rotate the polarization vector by the angle [2]:

$$d\Phi = \lambda^2 \frac{e^3}{8\pi^2 m_e^2}\, n_e\, \mathbf{B}\cdot\hat{\mathbf{n}}\, a\, d\eta\,, \tag{8}$$

where $\hat{\mathbf{n}}$ indicates the photon propagation direction, a is the scale factor, λ is the photon wavelength and n_e is the number density of free electrons. In the assumption of flux conservation, the quantity $\lambda^2\mathbf{B}$ is time independent and thus one can substitute $\lambda \to \lambda_0$ and $\mathbf{B} \to \mathbf{B}_0$ into equation (8).

The Faraday rotation is proportional to the number density of free electrons, which evolves because of the Universe expansion and because of recombination and reionization phenomena. It is useful to describe these variations in terms of their contribution to the photon optical depth. Following (Kosowsky & Loeb 1996), we introduce the differential

[2]Here and in the following, we use the Heaviside-Lorentz electromagnetic units in which the fine structure constant is $\alpha = e^2/4\pi$. In the previous version of this paper and in the version accepted for publication in ApJ, pre-factors in some formulae are incorrect. However, the final results are correct. We thank T. Kahniashvili for making us aware of this point.

optical depth $\dot{\tau} = d\tau/d\eta = n_e \sigma_T a$ where $\sigma_T = 8\pi\alpha^2/\,3m_e^2$ is the Thomson cross section. In terms of this quantity, one has

$$d\Phi = \frac{3}{4\pi e}\,\dot{\tau}(\eta)\,\lambda_0^2\,\mathbf{B}_0\cdot\hat{\mathbf{n}}\,d\eta\,. \qquad (9)$$

This expression has to be integrated along the photon path, starting from the photon last scatter η until the present time η_0. One obtains

$$\Delta\Phi(\eta) = \lambda_0^2\,\varrho(\eta)\,, \qquad (10)$$

where $\varrho(\eta)$ is given by

$$\varrho(\eta) = \frac{3}{4\pi e}\int_\eta^{\eta_0} d\eta'\;\dot{\tau}(\eta')\,\mathbf{B}_0(\hat{\mathbf{n}}(\eta'-\eta_0))\cdot\hat{\mathbf{n}}\,. \qquad (11)$$

In the previous expression we have assumed that the observer is at the origin of our reference frame and $\hat{\mathbf{n}}(\eta'-\eta_0)$ describes the photon space trajectory as a function of the conformal time η.

As a final step, one takes into account the fact that photons from a given direction last scattered at different times η. One introduces, then, the visibility function $g(\eta) = \dot{\tau}(\eta)\exp(-\tau(\eta))$, which gives the probability that a photon observed at η_0 last scattered within $d\eta$ of a given η, and calculates

$$\mathrm{RM} = \int_0^{\eta_0} d\eta\,g(\eta)\,\varrho(\eta) = \frac{3}{4\pi e}\int_0^{\eta_0} d\eta\,g(\eta)\int_\eta^{\eta_0} d\eta'\,\dot{\tau}(\eta')\,\mathbf{B}_0(\hat{\mathbf{n}}(\eta'-\eta_0))\cdot\hat{\mathbf{n}}\,. \qquad (12)$$

It is straightforward to recast Eq. (12) into the form

$$\mathrm{RM} = \frac{3}{4\pi e}\int_0^{\eta_0} d\eta' g(\eta')\,\mathbf{B}_0(\hat{\mathbf{n}}(\eta'-\eta_0))\cdot\hat{\mathbf{n}}\,, \qquad (13)$$

which gives the wavelength-independent Faraday rotation measure RM in terms only of $g(\eta)$ and of the properties of the magnetic field $\mathbf{B}_0$.

In principle, by measuring the CMBR polarization at different wavelengths with a suitable accuracy, one should be able to determine the Faraday RM as a function of the observation direction and, thus, to determine the correlator

$$RR'(\theta) = \langle \mathrm{RM}(\hat{\mathbf{n}})\mathrm{RM}(\hat{\mathbf{m}})\rangle\,, \qquad (14)$$

where $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ are two directions on the sky and $\cos\theta = \hat{\mathbf{n}}\cdot\hat{\mathbf{m}}$. This correlator depends on the properties of the magnetic field and on the ionization history of the Universe, according to

$$RR'(\theta) = \left(\frac{3}{4\pi e}\right)^2 \int d\eta\,g(\eta)\int d\eta' g(\eta')\,\langle(\mathbf{B}_0(\Delta\eta\,\hat{\mathbf{n}})\cdot\hat{\mathbf{n}})(\mathbf{B}_0(\Delta\eta'\,\hat{\mathbf{m}})\cdot\hat{\mathbf{m}})\rangle\,, \qquad (15)$$

where $\Delta\eta = \eta - \eta_0$ and $\Delta\eta' = \eta' - \eta_0$. The last terms in the previous expression can be expressed in Fourier space as

$$\begin{aligned}\langle(\mathbf{B}_0\cdot\hat{\mathbf{n}})(\mathbf{B}_0\cdot\hat{\mathbf{m}})\rangle \;=\; & \frac{1}{2(2\pi)^3}\int d^3\mathbf{k}\;\Big\{\Big[(\hat{\mathbf{n}}\cdot\hat{\mathbf{m}}) - (\hat{\mathbf{n}}\cdot\hat{\mathbf{k}})(\hat{\mathbf{m}}\cdot\hat{\mathbf{k}})\Big]\,S(k) \\ & +\; i\Big[(\hat{\mathbf{n}}\times\hat{\mathbf{m}})\cdot\hat{\mathbf{k}}\Big]\,A(k)\Big\}\exp\left[-i\mathbf{k}\cdot(\hat{\mathbf{n}}\Delta\eta - \hat{\mathbf{m}}\Delta\eta')\right]. \qquad (16)\end{aligned}$$

We remind the reader that $S(k)$ and $A(k)$ are, respectively, symmetric and helical terms of the magnetic field correlation tensor.

In principle, in order to calculate rigorously the Faraday RM correlator $RR'(\theta)$, one should solve the radiative transfer equations for the microwave background polarization in the presence of a magnetic field. The exact value of the rotation measure is, in fact, sensitive to the growth history of the CMBR polarization through the surface of the last scattering. This was done, *e.g.*, in (Kosowsky & Loeb 1996) where the quantity $RR'(0)$ was calculated in the limit of small magnetic fields (*i.e.* $B_0 \sim 10^{-9}$G), showing that, in this limit, the simple approach described above gives reasonably accurate results. In this paper, we use Eqs. (15,16), which has the advantage of simplicity and allows us to calculate analytically the angular dependence of RR'. This allows, in turn, to discuss in a transparent way the dependence of the Faraday RM maps on the magnetic field parameters, the effects of finite detector angular resolution and a possible role of helicity in RR'.

In this respect, it was suggested by Pogosian *et al.* (2002, 2003) that the dependence of RR' on $A(k)$ can be used to extract information on the helical part of the magnetic field. However, one can show that the helical part of the magnetic field does not contribute to RR' at any angle θ [3]. This is due to the fact that its contribution to the magnetic field correlator $\langle(\mathbf{B}_0\cdot\hat{\mathbf{n}})(\mathbf{B}_0\cdot\hat{\mathbf{m}})\rangle$, given by

$$\langle(\mathbf{B}_0\cdot\hat{\mathbf{n}})(\mathbf{B}_0\cdot\hat{\mathbf{m}})\rangle_{\rm hel} \equiv \frac{i}{2(2\pi)^3}\int d^3\mathbf{k}\left(\hat{\mathbf{n}}\times\hat{\mathbf{m}}\cdot\hat{\mathbf{k}}\right)A(k)\exp[-i\mathbf{k}\cdot(\hat{\mathbf{n}}\Delta\eta - \hat{\mathbf{m}}\Delta\eta')], \qquad (17)$$

is zero for any chosen directions $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$. Indeed, by decomposing $\mathbf{k}$ in terms of $\mathbf{k}_\perp$ and $\mathbf{k}_\parallel$, where $\mathbf{k}_\perp$ and $\mathbf{k}_\parallel$ are perpendicular and parallel, respectively, to the plane containing the vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$, we get

$$\langle(\mathbf{B}_0\cdot\hat{\mathbf{n}})(\mathbf{B}_0\cdot\hat{\mathbf{m}})\rangle_{\rm hel} = \frac{i}{2(2\pi)^3}\int d^3\mathbf{k}\left(\hat{\mathbf{n}}\times\hat{\mathbf{m}}\cdot\frac{\mathbf{k}_\perp}{k}\right)A(|\mathbf{k}_\perp + \mathbf{k}_\parallel|)\exp\big[-i\mathbf{k}_\parallel\cdot(\hat{\mathbf{n}}\Delta\eta - \hat{\mathbf{m}}\Delta\eta')\big]. \qquad (18)$$

[3]For $\theta = 0$ this follows simply from the fact that helicity affects only off-diagonal terms of the magnetic field autocorrelation function, as noted in (Enßlin & Vogt 2003).

One immediately sees that the integrand is odd in $\mathbf{k}_\perp$, and thus the integral has to vanish. All this shows that CMBR Faraday RM maps cannot give any information on the helicity of primordial magnetic fields [4].

In conclusion, Faraday RM maps of the CMBR only depends on the symmetric part of the magnetic field and we have

$$RR'(\theta) = \left(\frac{3}{4\pi e}\right)^2 \int d\eta\, g(\eta) \int d\eta'\, g(\eta') \langle (\mathbf{B}_0(\Delta\eta\hat{\mathbf{n}}) \cdot \hat{\mathbf{n}})(\mathbf{B}_0(\Delta\eta'\hat{\mathbf{m}}) \cdot \hat{\mathbf{m}})\rangle_{\rm symm} . \tag{19}$$

The magnetic field correlator can be expressed as (Kolatt 1997)

$$\langle (\mathbf{B}_0 \cdot \hat{\mathbf{n}})(\mathbf{B}_0 \cdot \hat{\mathbf{m}})\rangle_{\rm symm} = \left[(\hat{\mathbf{n}} \cdot \hat{\mathbf{m}})C_\perp(r) + (\hat{\mathbf{n}} \cdot \frac{\mathbf{r}}{r})(\hat{\mathbf{m}} \cdot \frac{\mathbf{r}}{r})(C_\parallel(r) - C_\perp(r))\right], \tag{20}$$

where $\mathbf{r} = \hat{\mathbf{n}}\Delta\eta - \hat{\mathbf{m}}\Delta\eta'$, and

$$C_\perp(r) \;=\; \frac{2}{3(2\pi)^3} \int_0^\infty dk\, \mathcal{E}_{\rm B}(k) \left[j_0(kr) - \frac{1}{2} j_2(kr)\right], \tag{21}$$

$$C_\parallel(r) \;=\; \frac{2}{3(2\pi)^3} \int_0^\infty dk\, \mathcal{E}_{\rm B}(k) \left[j_0(kr) + j_2(kr)\right], \tag{22}$$

where $j_i(x)$ are the spherical Bessel functions of the i^{th} order.

4. Results

The bulk of the Faraday rotation is generated close to $\eta_{\rm dec}$ where the photon visibility function $g(\eta)$ is maximal. If the correlation length ξ of the magnetic field is much larger than the thickness of last scattering surface $\delta\eta_{\rm dec} \sim 10$ Mpc (Spergel *et al.* 2003) (*i.e.* the distance travelled by a photon during the period of time in which $g(\eta)$ is sizeably different from zero), one can approximate the visibility function with delta function $g(\eta) = \delta(\eta - \eta_{\rm dec})$ and extract $\langle (\mathbf{B}_0 \cdot \mathbf{n})(\mathbf{B}_0 \cdot \mathbf{m})\rangle_{\rm symm}$ from the two integrals in Eq. (19). As a final result, one obtains

$$RR'(\theta) = \left(\frac{3}{4\pi e}\right)^2 \left\{\hat{\mathbf{n}} \cdot \hat{\mathbf{m}}\, C_\perp(r_{\rm dec}) + (\hat{\mathbf{n}} \cdot \frac{\mathbf{r}_{\rm dec}}{r_{\rm dec}})(\hat{\mathbf{m}} \cdot \frac{\mathbf{r}_{\rm dec}}{r_{\rm dec}}) \left[C_\parallel(r_{\rm dec}) - C_\perp(r_{\rm dec})\right]\right\}, \tag{23}$$

where $\hat{\mathbf{n}} \cdot \hat{\mathbf{m}} = \cos\theta$ and $\mathbf{r}_{\rm dec} = (\hat{\mathbf{n}} - \hat{\mathbf{m}})(\eta_{\rm dec} - \eta_0)$.

[4]During the writing of this paper, we learned from (Kahniashvili 2004) that Kosowsky *et al.* are also studying this problem with similar conclusions.

For the case $\theta = 0$, expression (23) can be easily evaluated. We have

$$RR'(0) = \frac{6}{(2\pi)^3(4\pi e)^2}\int dk\,\mathcal{E}_{\rm B}(k) = \frac{3}{(4\pi e)^2}\,\overline{B}_0^2\,, \tag{24}$$

where $\overline{B}_0$ is the average magnetic field, defined by

$$\overline{B}_0^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\mathbf{k}'}{(2\pi)^3}\langle B_0(\mathbf{k})\cdot B_0(\mathbf{k}')\rangle \;=\; \frac{2}{(2\pi)^3}E_B. \tag{25}$$

This result essentially coincides with the result of (Kosowsky & Loeb 1996) and corresponds to the average rotation of the CMBR polarization approximately equal to

$$\Phi = RR'(0)^{1/2}\lambda_0^2 \;\simeq\; 1.3^o\left(\frac{B_0}{10^{-9}\mathrm{Gauss}}\right)\left(\frac{\nu_0}{30\mathrm{GHz}}\right)^{-2}. \tag{26}$$

For $\theta \neq 0$, the situation is slightly more complicated. After some algebra we obtain

$$RR'(\theta) = \left(\frac{3}{4\pi e}\right)^2\left[\,C_\perp(r_{\rm dec})\cos^2(\theta/2)\;-\;C_\parallel(r_{\rm dec})\sin^2(\theta/2)\,\right], \tag{27}$$

where $r_{\rm dec} = 2(\eta_0-\eta_{\rm dec})\sin(\theta/2)$. However, in a realistic situation, the correlation length ξ of the magnetic field is much smaller than the distance to the last scattering surface, $\eta_0 - \eta_{\rm dec}$. This permits us to consider the limit of small observation angles (*i.e.* $\theta \ll 1$), in which the second term in the right-hand side of Eq. (27) becomes negligible and one obtains

$$\frac{RR'(\theta)}{RR'(0)} = \frac{C_\perp(\theta(\eta_0-\eta_{\rm dec}))}{C_\perp(0)}\,. \tag{28}$$

We remark that this expression is quite natural if we consider the physical meaning of the function $C_\perp(r)$. It describes the variation of the correlator $\langle B_i(0)B_i(\mathbf{r})\rangle = C_\perp(r)$ of the magnetic field components in a fixed direction (which in our case is the photon propagation direction), when $\mathbf{r}$ moves in a plane perpendicular to that direction (*i.e.* on the last scattering surface).

To obtain an explicit expression for $RR'(\theta)$ one has now to specify $\mathcal{E}_{\rm B}(k) = 2\pi k^2 S(k)$ in Eq. (21). If $S(k)$ can be described by the simple functions given by Eq. (5), the integration can be performed analytically. For the case $n_{\rm S} = 2$ we get

$$\frac{RR'(\theta)}{RR'(0)} = e^{-\chi^2\theta^2}\left(1-\chi^2\theta^2\right), \tag{29}$$

where $\chi = K(\eta_0-\eta_{\rm dec})/2$. For different n_S values one obtains quite similar expressions, of the kind $\exp(-\chi^2\theta^2)P(\chi^2\theta^2)$, where $P(\chi^2\theta^2)$ are higher order polynomials in the $\chi^2\theta^2$

variable [in the Appendix we present explicit expressions of $C_\perp(r)$ and $C_\parallel(r)$ for arbitrary values of n_S]. Expression (29) can be explicitly written as a function of the magnetic field correlation length ξ, which for $n_S = 2$ is $\xi = 8\sqrt{\pi}/3K$. For different n_S values, one has to take into account that the relation between ξ and K depends on the chosen spectral index, as it is described in Eq. (6).

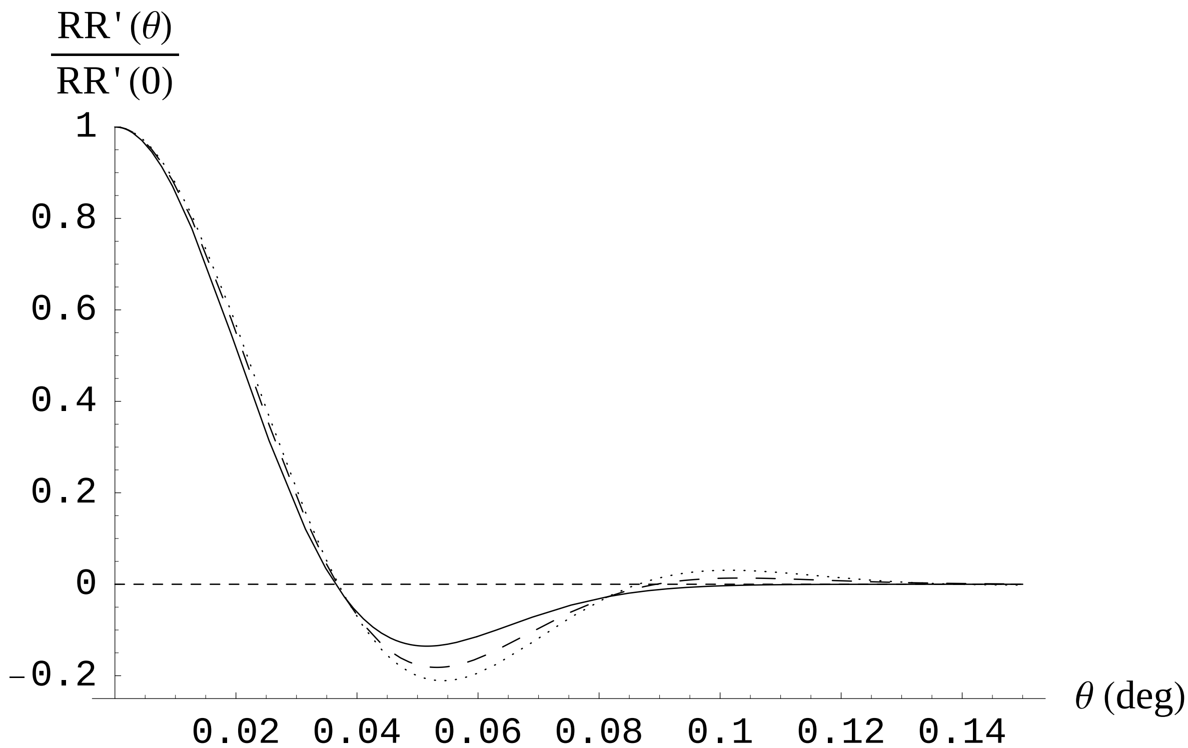

Fig. 1.— Faraday rotation measure correlation $RR'(\theta)$ as a function of the separation angle θ. The three lines correspond to the magnetic field spectral index $n_S = 2$ (*solid line*), $n_S = 4$ (*dashed line*) and $n_S = 6$ (*dotted line*). The correlation length of the magnetic field is $\xi = 20$ Mpc.

In Fig. 1 we show RR' as a function of the angle θ (normalized to its value at $\theta = 0$) for various choices of the spectral index n_S and for a fixed correlation length $\xi = 20$ Mpc. The three lines correspond to $n_S = 2, 4$, and 6. One can see that the function $RR'(\theta)$ has quite peculiar behavior. In particular, in all cases, there are angles that correspond to negative values of RR'. This reflects the behavior of the magnetic field at the last scattering surface. Namely, it is due to the fact that the correlator $\langle B_\gamma(0) B_\gamma(r)\rangle \simeq C_\perp(r)$ (where γ indicates the photons propagation direction) on the surface of the last scatter becomes negative. In other words, the magnetic field components B_γ in different regions of the last scattering surface can be anticorrelated.

From the point of view of observations, it is important to note that the angular behavior of $RR'(\theta)$ is only marginally dependent on the n_S value. In particular, the point at which $RR'(\theta)$ vanishes practically coincides in all three cases. On one hand, this indicates that the observation of $RR'(\theta)$ could provide information on the correlation length ξ that is essentially independent of the parameter n_S. On the other hand, it indicates that a large sensitivity is required to discriminate among different n_S values.

The results presented above are obtained neglecting the last scattering surface thickness and assuming that the detector has a perfect angular response. These approximations are correct if the magnetic field correlation length ξ is much larger than $\delta\eta_{\rm dec}$ and if, at the same time, $\theta_\xi \gg \sigma$, where σ is the detector angular resolution and $\theta_\xi = \xi/(\eta_0 - \eta_{\rm dec})$ represents the angular dimension of a correlation domain on the surface of the last scatter. Both effects of $\delta\eta_{\rm dec} \neq 0$ and of $\sigma \neq 0$ reduce the calculated value for RR'. In this sense, Eq. (24) should be regarded as an upper limit for the true $RR'(0)$ value.

In order to estimate the role of $\delta\eta_{\rm dec}$ in the calculation of $RR'(0)$, one can approximate the behavior of $g(\eta)$ around the decoupling with $g(\eta) = (1/\sqrt{2\pi}\delta_{\rm dec}) \exp\left[-(\eta - \eta_{\rm dec})^2/2\delta\eta_{\rm dec}^2\right]$, which is a Gaussian peaked at $\eta_{\rm dec}$ with a characteristic width equal to $\delta\eta_{\rm dec}$. Introducing this function into Eq. (19) one can calculate explicitly the ratio between the rotation measure obtained taking into account the thickness of the last scattering surface, $\langle RR'(0)\rangle_{\delta\eta_{\rm dec}}$, and the "ideal" value, $RR'(0)$, given by Eq. (24). One obtains

$$\frac{\langle RR'(0)\rangle_{\delta\eta_{\rm dec}}}{RR(0)} = \frac{1}{2\sqrt{\pi}\delta\eta_{\rm dec}} \int_{-\infty}^{\infty} d\eta' \exp\left(-\frac{\eta'^2}{4\delta\eta_{\rm dec}^2}\right) \frac{C_\parallel(\eta')}{C_\parallel(0)} \,. \tag{30}$$

One should note that the thickness of the last scattering surface introduces an average (along a line) of the function $C_\parallel(r)$. This is easily explained if one considers the physical meaning of this function. It describes (see Eq. (20)) the variations of the correlators $\langle B_i(0)B_i(\mathbf{r})\rangle$ of the magnetic field components in a fixed direction (which in our case is the photon propagation direction), when $\mathbf{r}$ moves parallel to this direction (*i.e.* across the last scattering surface). For the specific case $n_S = 2$, we have

$$\frac{C_\parallel(\eta')}{C_\parallel(0)} = e^{-\frac{K^2\eta'^2}{4}}, \tag{31}$$

which gives

$$\frac{\langle RR'(0)\rangle_{\delta\eta_{\rm dec}}}{RR(0)} = \left[1 + \frac{64\pi}{9}\left(\frac{\delta\eta_{\rm dec}}{\xi}\right)^2\right]^{-1/2} . \tag{32}$$

For different values of the n_S parameters, we obtain more complicated expressions (which can be calculated analytically; see the Appendix). For all the values of the n_S parameter,

however, the ratio $\langle RR'(0)\rangle_{\delta\eta_{\rm dec}}/RR'(0)$, for $\delta\eta_{\rm dec} \gg \xi$, always goes to zero as $(\delta\eta_{\rm dec}/\xi)^{-1}$. In conclusion, since Φ is proportional to $(RR')^{1/2}$ (see Eq. (26), we find that the average rotation angle of the CMBR photons is reduced with respect to Eq. (24) as $N^{-1/2}$, where $N = \delta_{\rm dec}/\xi$ is the number of correlation domains in the length $\delta\eta_{\rm dec}$.

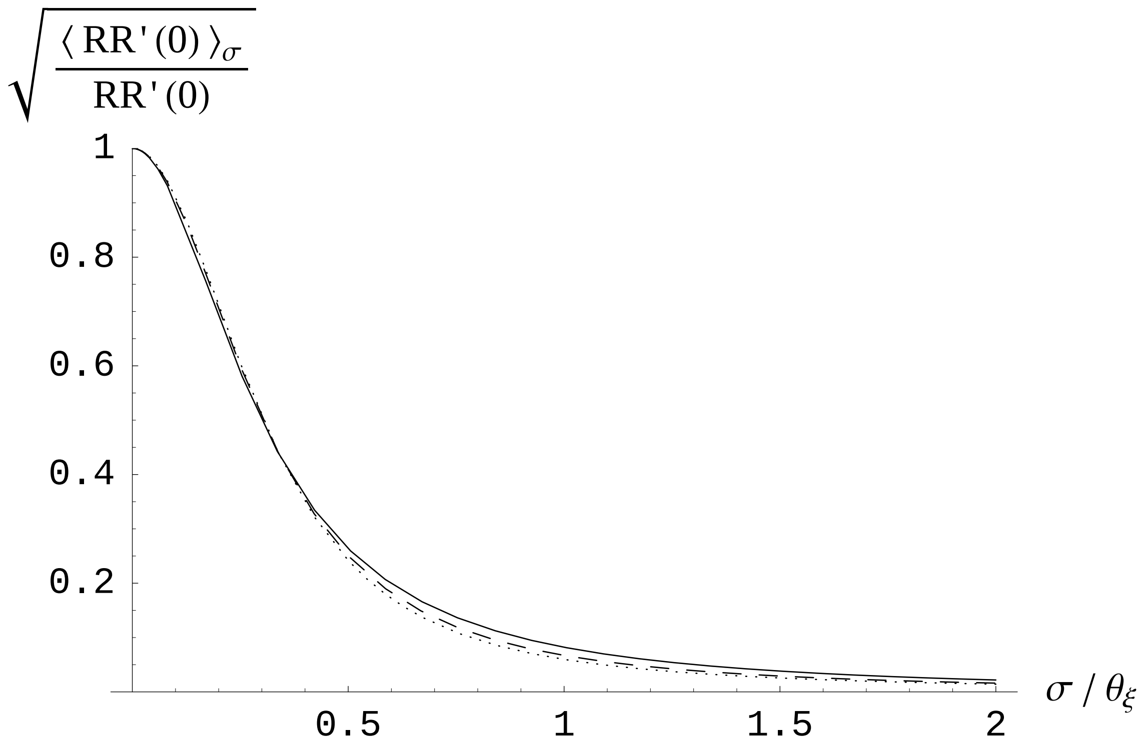

Fig. 2.— Effect of the detector angular resolution σ on the CMBR Faraday rotation measure. The angle θ_ξ represents the angular dimension of a magnetic field correlation domain on the surface of the last scattering. The three lines correspond to the magnetic field spectral index $n_S = 2$ (*solid line*), $n_S = 4$ (*dashed line*) and $n_S = 6$ (*dotted line*).

A similar exercise can be done to include the effect of finite detector angular resolution. Modelling the detector angular response with a Gaussian of angular width σ (Kolb & Turner 1990), one obtains

$$\frac{\langle RR'(0)\rangle_\sigma}{RR(0)} = \frac{1}{\sigma^2}\int_0^\infty d\theta\,\theta\exp\left(-\frac{\theta^2}{2\sigma^2}\right)\frac{C_\perp(\theta(\eta_0-\eta_{\rm dec}))}{C_\perp(0)}\,, \tag{33}$$

where $\langle RR'(0)\rangle_\sigma$ is the rotation correlator obtained with an account of the detector angular resolution. We have assumed here that $\sigma \ll 1$. In the case $n_S = 2$, one obtains

$$\frac{\langle RR'(0)\rangle_\sigma}{RR(0)} = \left[1+\frac{32\pi}{9}\left(\frac{\sigma}{\theta_\xi}\right)^2\right]^{-2}, \tag{34}$$

where the angle $\theta_\xi = \xi/(\eta_0 - \eta_{\rm dec})$ represents the angular dimension of a correlation domain on the surface of the last scattering. For other n_S values, we obtain different expressions (see the Appendix) which, for $\sigma \gg \theta_\xi$, always go to zero as $(\theta_\xi/\sigma)^{-4}$. Thus, the rotation angle Φ is reduced because of effect of the angular resolution as $(\sigma/\theta_\xi)^{-2}$. We remark that this behavior could introduce severe limitations to the actual possibility of observing the Faraday rotation of the CMBR polarization. In particular, in Fig.2, we show $[\langle RR'(0)\rangle_\sigma / RR'(0)]^{1/2}$ as a function of the ratio σ/θ_ξ for different n_S values. One sees that even for small values of σ/θ_ξ the reduction of the effect can be substantial.

We point out, finally, that in contrast to the previous case, Φ is reduced as N_σ^{-1} (and not as $N_\sigma^{-1/2}$), where $N_\sigma = (\sigma/\theta_\xi)^2$ is the number of correlation domains in a (two-dimensional) region at the last scattering surface of angular dimension σ. This is due to the peculiar behavior of the $C_\perp$ functions or, in other words, to the fact that magnetic field components in different regions of the last scattering surface can be anticorrelated.

5. Conclusions

In this paper we have analyzed the Faraday rotation of the CMBR polarization induced by a primordial stochastic magnetic field. In particular, we have calculated the correlation between the Faraday rotation measures, $RR'(\theta)$, as a function of the separation angle between observation directions, θ, for different assumptions about the magnetic field spectral index and correlation length. Here we summarize here our main results.

i) The helical part of the magnetic field does not contribute to $RR'(\theta)$ at any angle θ. This means that Faraday RM maps can provide information only on the symmetric part of the magnetic field.

ii) In the approach described in Sec. 3, neglecting the last scattering surface thickness and the detector angular resolution, the Faraday RM maps $RR'(\theta)$ can be calculated analytically. We have provided an analytic expression for $RR'(\theta)$ for arbitrary values of the magnetic field spectral index and correlation length.

iii) The last scattering surface scale thickness $\delta\eta_{\rm dec}$ reduces the Faraday rotation angle with respect to the simple estimate given by Eq. (26). In the limit $\delta\eta_{\rm dec} \gg \xi$, where ξ is the magnetic field correlation length, the rotation angle is reduced as $(\delta\eta_{\rm dec}/\xi)^{-1/2}$ (see Eq. (32)).

iv) The detector angular resolution σ could drastically reduce the possibility of observing the Faraday rotation. In the limit $\sigma \gg \theta_\xi$, where θ_ξ is the angular dimension of a magnetic

field correlation domain, the rotation angle is reduced with respect to Eq. (26) as $(\sigma/\theta_\xi)^{-2}$ (see Eq. (34)). As shown in Fig. 2, the reduction can be substantial even for small values of σ/θ_ξ.

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A. The $C_\perp(r)$ and $C_\parallel(r)$ correlators

In this appendix, we calculate the correlators $C_\perp(r)$ and $C_\parallel(r)$ defined by Eqs. (21) and (22), for the functional form of the magnetic power spectrum given by Eq. (5). To this end, we introduce the function

$$L_{n_S}(\alpha) = \frac{\int_0^\infty dk k^{n_s+2} e^{-\alpha k^2/K^2} \left[j_0(kr) - j_2(kr)/2 \right]}{\int_0^\infty dk k^{n_s+2} e^{-k^2/K^2}} \,, \tag{A1}$$

related to $C_\perp(r)$ by

$$L_{n_S}(0) = \frac{C_\perp(r)}{C_\perp(0)} \,. \tag{A2}$$

It is straightforward to obtain the recursion formula for $L_{n_S}(\alpha)$

$$L_{n_S}(\alpha) = A_m \frac{\partial^m}{\partial \alpha^m} L_2(\alpha) \,, \tag{A3}$$

where the "generating function" $L_2(\alpha)$ is given by

$$L_2(\alpha) = \frac{e^{-\kappa^2 r^2/\alpha}}{\alpha^{5/2}} \left(1 - \frac{\kappa^2 r^2}{\alpha} \right), \tag{A4}$$

and

$$m = \frac{n_S}{2} - 1, \quad \kappa = \frac{K}{2}, \quad A_n = \frac{3\,(-1)^n\, 2^n}{(2n+3)!!} \tag{A5}$$

(we remember that $n_S \geq 2$ is an even natural number).
Taking into account Eqs. (A3)-(A4) we cast Eq. (A2) in the form

$$\frac{C_\perp(r)}{C_\perp(0)} = e^{-\kappa^2 r^2} P_{\frac{n_S}{2}-1}(\kappa^2 r^2) \,, \tag{A6}$$

where $P_n(x)$ turn out to be polynomials of $(n+1)$st degree defined by

$$P_n(x) = A_n\, e^x \frac{\partial^n}{\partial \alpha^n} \frac{e^{-x/\alpha}}{\alpha^{5/2}} \left(1 - \frac{x}{\alpha} \right) |_{\alpha=1} \,, \quad n = 0, 1, 2, ... \tag{A7}$$

The first three P_n polynomials are

$$\begin{aligned} P_0(x) &= 1 - x, \\ P_1(x) &= 1 - \frac{9}{5}\, x + \frac{2}{5}\, x^2, \\ P_2(x) &= 1 - \frac{13}{5}\, x + \frac{8}{7}\, x^2 - \frac{4}{35}\, x^3. \end{aligned} \tag{A8}$$

Following the same procedure, we find for $C_\parallel(r)$ the expression

$$\frac{C_\parallel(r)}{C_\parallel(0)} = e^{-\kappa^2 r^2} Q_{\frac{n_s}{2}-1}(\kappa^2 r^2)\,, \tag{A9}$$

where $Q_n(x)$ are polynomials of nth degree given by

$$Q_n(x) = A_n\, e^x \frac{\partial^n}{\partial\alpha^n} \frac{e^{-x/\alpha}}{\alpha^{5/2}}|_{\alpha=1}\,, \quad n = 0, 1, 2, ..., \tag{A10}$$

where κ and A_n are the same as in Eq. (A5). We give the expressions for the first three Q_n polynomials:

$$\begin{aligned} Q_0(x) &= 1, \\ Q_1(x) &= 1 - \frac{2}{5}\,x, \\ Q_2(x) &= 1 - \frac{4}{5}\,x + \frac{4}{35}\,x^2. \end{aligned} \tag{A11}$$

Now, having the expressions for $C_\perp(r)$ and $C_\parallel(r)$, we can easily calculate the correlators defined by Eqs. (30) and (34). Taking into account Eqs. (A9)-(A10), we cast Eq. (30) in the form

$$\frac{\langle RR'(0)\rangle_{\delta\eta_{\rm dec}}}{RR'(0)} = A_m \frac{\partial^m}{\partial\alpha^m} \frac{1}{\alpha^2\sqrt{\alpha + (K\delta\eta_{\rm dec})^2}}\,|_{\alpha=1}\,, \tag{A12}$$

while, inserting Eqs. (A6)-(A7) into Eq. (34), we get

$$\frac{\langle RR'(0)\rangle_\sigma}{RR'(0)} = A_m \frac{\partial^m}{\partial\alpha^m} \frac{1}{\sqrt{\alpha}\,(\alpha + 2\chi^2\sigma^2)^2}\,|_{\alpha=1}\,, \tag{A13}$$

where we remember that $\chi = K(\eta_0 - \eta_{\rm dec})/2$, and m is given by Eq. (A5).
Finally, we give the asymptotic expressions

$$\delta\eta_{\rm dec} \gg \xi: \qquad \frac{\langle RR'(0)\rangle_{\delta\eta_{\rm dec}}}{RR'(0)} = \frac{3}{8\sqrt{\pi}} \left(\frac{\delta\eta_{\rm dec}}{\xi}\right)^{-1}, \tag{A14}$$

$$\sigma \gg \theta_\xi: \qquad \frac{\langle RR'(0)\rangle_\sigma}{RR'(0)} = C_m \left(\frac{\sigma}{\theta_\xi}\right)^{-4}, \tag{A15}$$

where we remember that ξ is the correlation length (see Eq. (6)), the coefficients C_m are

$$C_m = \frac{3}{(2\pi)^4} \frac{\Gamma(m+1/2)\,[\,\Gamma(m+5/2)\,]^3}{[\,\Gamma(m+2)\,]^4}\,, \tag{A16}$$

and $\theta_\xi = \xi/(\eta_0 - \eta_{\rm dec})$ is the angular dimension of a correlation domain on the surface of the last scatter.

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